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PROPAGATION OF ENERGY IN A SYSTEM OF JOINED ELASTIC HALF-STRIPS OF DIFFERENT THICKNESS†

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Using a previously proposed method to allow for singularities of the solution at corner points of a compound elastic waveguide, which employs the generalized orthogonality property of normal modes, an analysis is presented of the frequency dependence of the energy reflection and transmission coefficients at the vertical joining line of the half-strips, and of the structure of the energy fluxes in its neighbourhood in the so-called energy capture range. In this situation there is a phenomenon of energy vortices partly overlapping the energy flux, leading to an increase in the energy reflection coefficient.

Methods of solving problems encountered in the transmission of elastic wave energy through the interface of a compound waveguide have seen considerable development, progressing from the collocation method [1] to more accurate methods, based on using generalized orthogonality relations of homogeneous solutions [1]—a technique first employed in [3, 4] and generalized to waveguides with stepped interface and vertical defects in [5]. However, the solution of the problem of a vertical interface in a waveguide has been complicated by the fact that, when certain properties of the materials occur in combination, or when there are defects at the interface, the stresses at the corner points involve singularities of the boundary-condition type, and the series representing the stresses diverge over finite intervals at the interface ([6–9], and so on).

In the case of an elastic half-strip, various ways have been devised to overcome these difficulties, ranging from the use of generalized sums of divergent series [9] to the explicit determination of the singularity [8, 10], or by allowing for the asymptotic behaviour of the unknowns of the expansion dictated by the index of the singularity [11]. A generalization of the method of determining the singularity to the case of compound waveguides, which essentially uses the generalized orthogonality property of the normal modes, was proposed in [12]. In this paper, based on that approach, we will analyse the transmission and reflection of energy, averaged over an oscillation period, in a compound stepped waveguide with a free surface. We will consider both the frequency dependence of the transmission and reflection coefficients, and the structure of the energy fluxes near the line of the end face.

1. We will consider steady oscillations of a free compound elastic waveguide consisting of two joined half-strips $-\infty < x \leqslant 0$, $0 \leqslant z \leqslant h_1$ (the first medium) and $0 \leqslant x < \infty$, $0 \leqslant z \leqslant h_2$ (the second medium) with different properties: density ρ_j, Lamé coefficients λ_j and μ_j and Poisson's ratio ν_j, where $j = 1, 2$ represents the first or second medium, respectively. To fix our ideas, let $h_1 > h_2$. The source of the oscillations are travelling waves $\mathbf{u}_0 e^{-i\omega t}$ arriving either from a load $\mathbf{q}(x)e^{-i\omega t}$ applied to the surface $z = h_1$ of the left half-plane in a domain Ω, or from infinity. The factor $e^{-i\omega t}$ will henceforth be omitted.

The outer surface of the waveguide is stress-free

$$\mathbf{T}_{n,j} = 0, \quad z = 0; \quad \mathbf{T}_{n,j} = \mathbf{q}(x), \quad z = h_j, \quad \mathbf{q}(x) = 0, \quad x \notin \Omega$$
$$\sigma_{x,1}(0,z) = \tau_{xz,1}(0,z) = 0, \quad h_2 \leqslant z \leqslant h_1$$

where $\mathbf{T}_{nj} = (\tau_{zxj}, \sigma_{zj})$ is the vector of stresses on the horizontal surface.

In addition, the following conditions are satisfied on joining line of the half-strips

$$\mathbf{u}_1(0,z) = \mathbf{u}_2(0,z), \quad 0 \leqslant z \leqslant h_2$$
$$\sigma_{x,1}(0,z) = \sigma_{x,2}(0,z), \quad \tau_{xz,1}(0,z) = \tau_{xz,2}(0,z), \quad 0 \leqslant z \leqslant h_2$$

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Let $\mathbf{u}_1 + \mathbf{u}_0$ denote the displacements in the first medium and $\mathbf{u}_2$ those in the second medium; $\mathbf{u}_0$ denotes the displacement field of the incident wave and $\mathbf{u}_1$ that of the reflected wave.

The displacement fields can be expanded in terms of normal waves (henceforth throughout summation is from 1 to ∞)

$$\mathbf{u}_1(x,z) = \sum_m t_{1m}\mathbf{a}_{1m}(z)e^{-i\alpha_{1m}x}, \quad \mathbf{u}_2(x,z) = \sum_m t_{2m}\mathbf{a}_{2m}(z)e^{-i\alpha_{2m}x} \tag{1.1}$$

$$\mathbf{a}_{jm}(z) = \frac{1}{\Delta_j'(\alpha_{jm})}\left\| \begin{matrix} (-1)^{j+1}\alpha_{jm}P_{jm}(z) \\ iR_{jm}(z) \end{matrix} \right\|$$

The eigenvectors $\mathbf{a}_{jm}$ are expressed in terms of the components of the symbol of Green's matrix of a free elastic layer $\alpha P_j(\alpha, z)/\Delta(\alpha)$, $R_j(\alpha, z)/\Delta(\alpha)$ (we are using the notation of [13]), $P_{jm}(z) = P_j(\alpha_{jm}, z)$, $R_{jm}(z) = R_j(\alpha_{jm}, z)$; the prime denotes differentiation with respect to z. The wave numbers α_{jm} are identical with the poles of the elements of Green's matrix (with zero of its denominator) and t_{jm} are the unknown expansion coefficients.

Series (1.1) contain both symmetric and antisymmetric modes: the wave for $m = 1$ is symmetric, the waves for $m = 2$ and 3 are antisymmetric, for $m = 4, 5$ symmetric, and so on.

In the analogous representations for the vector of stresses $\tau_j(x, z) = (\sigma_{x,j}, \tau_{xz,j})$, the vectors $\mathbf{a}_{jm}$ are replaced by

$$\mathbf{b}_{jm}(z) = \frac{1}{\Delta_j'(\alpha_{jm})}\left\| \begin{matrix} -i(\lambda_j + 2\mu_j)\alpha_{jm}^2 P_{jm}(z) + i\lambda_j R_{jm}'(z) \\ (-1)^{j+1}\mu_j\alpha_{jm}(P_{jm}'(z) + R_{jm}(z)) \end{matrix} \right\|$$

For the case of a wave arriving from infinity

$$\mathbf{u}_0 = \mathbf{a}_{0m}(z)e^{i\alpha_{1m}x}, \quad \tau_0 = \mathbf{b}_{0m}(z)e^{i\alpha_{1m}x} \tag{1.2}$$

where $m = 1, 2, \ldots$ is the number of the incident wave, $\mathbf{a}_{0m}$ differs from $\mathbf{a}_{1m}$ in the sign of the first component, and $\mathbf{b}_{0m}$ differs from $\mathbf{b}_{1m}$ in the sign of the second component.

Let us consider an auxiliary problem for isolated half-strips of equal thickness, on the assumption that the displacements $\mathbf{v}_j$ and stresses $\mathbf{t}_j$ ($j = 1, 2$) are prescribed at the ends, and a wave $\mathbf{u}_0$ propagates in the first medium. This problem is overdetermined.

If we assume that $\mathbf{v}_j$ and $\mathbf{t}_j$ belong to $L_2([0, h])$, the method described in [2] yields exact solutions of the auxiliary problem. When the stress has a singularity at the corner points $x = 0, z = 0$ and $z = h$, the series for $\tau_j(x, z)$ diverge in the neighbourhood of both points. However, this is of no significance when one is investigating energy propagation: it is sufficient that the boundary conditions hold in the mean-square sense.

To obtain a solution [2], multiply the boundary condition for the first component of the displacements by $b_{jk}^{(1)}$ (the first component of the vector $\mathbf{b}_{jk}$) and subtract the condition for the second component of the stresses multiplied by $a_{jk}^{(2)}$. The other conditions are transformed in the same way.

Using the generalized orthogonality conditions [2]

$$\int_0^h [a_{jm}^{(1)}b_{jk}^{(1)} - a_{jk}^{(2)}b_{jm}^{(2)}]dz = 0, \quad k \neq m$$

we obtain

$$t_{jm} = \frac{(-1)^p}{d_{jm}}[(v_j^{(l)}, b_{jm}^{(l)}) - (t_j^{(p)}, a_{jm}^{(p)}) - \delta_{j1}((u_0^{(l)}, b_{1m}^{(l)}) - (\tau_0^{(p)}, a_{1m}^{(p)}))] \tag{1.3}$$

$$d_{jm} = (a_{jm}^{(1)}, b_{jm}^{(1)}) - (a_{jm}^{(2)}, b_{jm}^{(2)}), \quad (f,g) = \int_0^h fg\,dz, \quad l = 1,2, \quad p = \begin{cases} 1, & l = 2, \\ 2, & l = 1. \end{cases}$$

Thus, the solution of the auxiliary problem requires only two boundary conditions; equating the right-hand sides of equalities (1.3) for different l, we obtain the "compatibility conditions" that the boundary conditions must satisfy if the auxiliary problem is to be solvable

$$(\mathbf{v}_j, \mathbf{b}_{jm}) - (\boldsymbol{\tau}_j, \mathbf{a}_{jm}) = \delta_{j1}[(\mathbf{u}_0, \mathbf{b}_{jm}) - (\boldsymbol{\tau}_0, \mathbf{a}_{jm})] \tag{1.4}$$

$$(\mathbf{f}, \mathbf{g}) = (f^{(1)}, g^{(1)}) + (f^{(2)}, g^{(2)})$$

Substituting the boundary conditions of the original problem $\mathbf{v}_1 = \mathbf{v}_2$, $\mathbf{t}_1 = \mathbf{t}_2$ for the case of half-strips of equal thickness ($h_1 = h_2 = h$) into (1.4), we obtain two independent infinite systems of linear algebraic equations for the unknowns t_{jm}

$$\left(\sum_m \mathbf{a}_{lm} t_{lm}, \mathbf{b}_{jn}\right) - \left(\sum_m \mathbf{b}_{lm} t_{lm}, \mathbf{a}_{jn}\right) = (-1)^l [(\mathbf{u}_0, \mathbf{b}_{jn}) - (\boldsymbol{\tau}_0, \mathbf{a}_{jn})] \tag{1.5}$$

$$j = 1, 2, \quad l = \begin{cases} 1, & j = 2, \\ 2, & j = 1, \end{cases} \quad n = 1, 2, \ldots$$

We need solve only one of systems (1.5), for $j = 1, l = 2$ or for $j = 2, l = 1$. The other unknowns may then be determined using formulae (1.3), i.e. the system splits into two independent subsystems, considerably reducing the volume of computations. In the general case when $h_1 \neq h_2$, Eqs (1.5), besides becoming slightly more complicated, are no longer independent for different j, and one then has a single system but of twice the dimensions.

To take the singularities of the stresses into account, one can separate out the factors with singularities in explicit form in the unknown stresses $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}$ and expand these stresses in a series of Jacobi polynomials

$$t^{(l)} = \sum_n P_n^{(l)} \chi_n^{(l)}, \quad 0 \leqslant z \leqslant h_2 \tag{1.6}$$

$$\mathbf{t} = 0, \quad h_2 \leqslant z \leqslant h_1$$

$$P_n^{(l)}(\zeta) = (1 - \zeta)^{\gamma_1} (1 + \zeta)^{\gamma_2} P_{i(n,l)}^{(\gamma_1 \gamma_2)}(\zeta), \quad \zeta = -2z / h_2 + 1, \quad 0 \leqslant z \leqslant h_2$$

where $\chi_n^{(l)}$ are unknown constants and $P_n^{(\gamma_1, \gamma_2)}(\zeta)$ are Jacobi polynomials; the order of the polynomials, $i = i(n, l)$, is chosen so that the evenness or oddness of each term in expansion (1.6) is the same as in the expansion in normal waves $b_{jn}^{(l)}$; γ_1 and γ_2 are the indices of the singularities of the stresses,† which have already been determined [14]. It is also very important that $\mathbf{t}$ is defined as zero on the free surface of the step.

Substituting these expansions into (1.3) and into the as yet unused boundary conditions $\mathbf{v}_1 = \mathbf{v}_2$ (for $h_1 \neq h_2$) we obtain an infinite system of four groups of linear algebraic equations for the four groups of unknowns t_{jm}, $\chi_m^{(l)}$ ($j, l = 1, 2$)

$$t_{jm} = \frac{(-1)^p}{d_{jm}} \left[\left(\sum_n a_{rn}^{(l)} t_{rn}, b_{jm}^{(l)} \right)_J + \delta_{j1} \left(\sum_n a_{jn}^{(l)} t_{jn}, b_{jm}^{(l)} \right)_T - \right.$$

$$\left. - \left(\sum_n P_n^{(p)} \chi_n^{(p)}, a_{jm}^{(p)} \right)_J - (-1)^r (u_0^{(l)}, b_{jm}^{(l)})_J + \delta_{j1} (\tau_0^{(p)}, a_{jm}^{(p)}) \right], \quad j, l = 1, 2,$$

$$p = \begin{cases} 1, & l = 2, \\ 2, & l = 1, \end{cases} \quad r = \begin{cases} 1, & j = 2, \\ 2, & j = 1, \end{cases} \quad n = 1, 2, \ldots,$$

$$(f, g)_J = \int_0^{h_2} fg dz, \quad (f, g)_T = \int_{h_2}^{h_1} fg dz.$$

2. The total energy flux, averaged over an oscillation period $T = 2\pi/\omega$, through a surface S is given by the expression

†GLUSHKOV, Ye. V. and GLUSHKOVA, N. V., On a singularity of the solution at corner points of compound elastic waveguides. Moscow, 1991. Deposited at VINITI 19.02.91, No. 824-V91.

$$E = \int_S e_n dS$$

where e_n is the projection of the energy flux density vector onto the normal **n** to S. For a cross-section of the waveguide with normal $\mathbf{n} = (0, 1)$

$$E_{x,j} = \int_0^{h_j} e_{x,j} dz, \quad e_{x,j} = \frac{\omega}{2} \operatorname{Im}(\tau_j, \mathbf{u}_j)$$

It has been shown [1] that for the joint of half-strips of equal thickness, in the frequency range where the number of travelling waves in the left half-strip exceeds that in the right half-strip, the interface effectively becomes a reflector—a phenomenon known as "energy capture". Figure 1 plots the energy reflection coefficient against frequency for various thicknesses of the left half-strip. It is obvious that an analogous phenomenon will occur for half-strips of different thicknesses. Outside the domain of energy capture, for the h_2 values shown in Fig. 1, the step at the interface has practically no effect on the energy transmission.

The parameters of the media for Fig. 1 were chosen to be those of [1] (in dimensionless form): $\mu_1 = 1$, $\mu_2 = 0.93480$, $\rho_1 = 1$, $\rho_2 = 0.6162$, $\nu_1 = 0.24$, $\nu_2 = 0.3$, $h_1 = 1$. The solid curve corresponds to the joint of half-strips of equal thickness, the dashed curve to the case $h_2 = 0.9h_1$ and the dash–dot curve to the case $h_2 = 0.85h_1$. The results for $h_1 = h_2$ agree with those in [1]. Another criterion for the correctness of the results was numerical verification of the energy balance and the boundary conditions on the joining line.

For half-strips of equal thickness, the computations allowed for two pairs of non-uniform waves. The error in the satisfaction of the boundary conditions at the joint was then of the order of a few percent. As it turned out, to investigate the energy flux in this case it was not at all necessary to take non-uniform waves into consideration: even if such waves were ignored, the reflection coefficient remained almost unchanged in value. Allowance for a singularity in the stresses when the half-strips are of equal thickness does not affect the values obtained for the energy flux. It only increases the accuracy with which the boundary conditions are satisfied—for combinations of media in which singularities occur. However, allowance for non-uniform waves and the structure of the stresses becomes very important when the half-strips area of different thickness: if the representation (1.6) is not used, the necessary accuracy cannot be achieved.

3. Energy streamlines are curves, the tangents to which at each point coincide in direction with the density vector of the total energy flux **e** averaged over one period. They are defined by the equation

$$d\mathbf{x} / ds = \mathbf{e}(s) / |\mathbf{e}|$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is the initial point and s is a natural parameter along the curve.

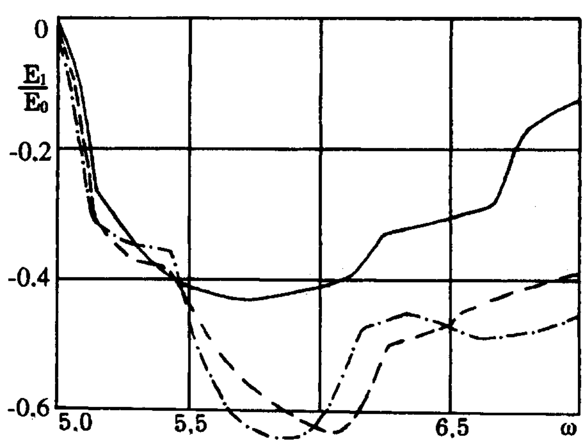

Fig. 1.

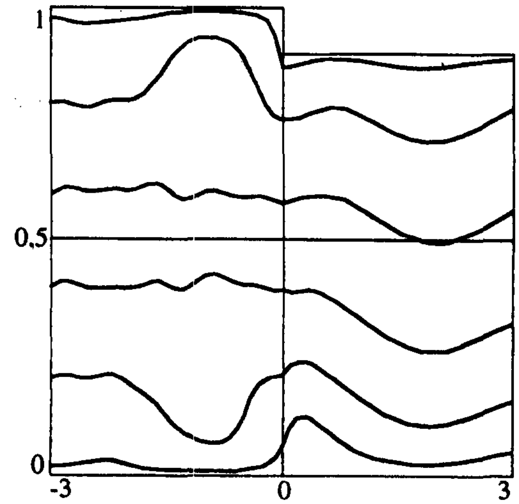

Fig. 2.

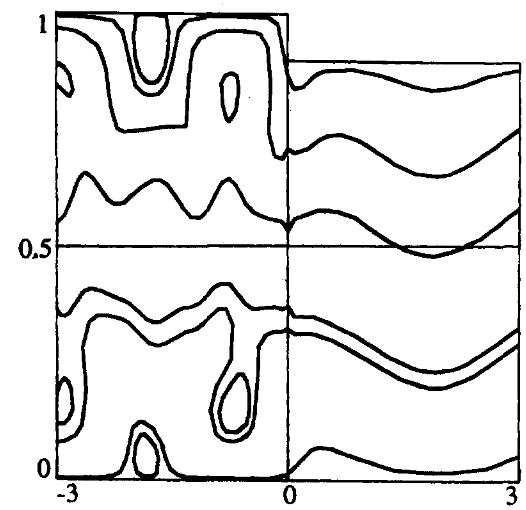

Fig. 3.

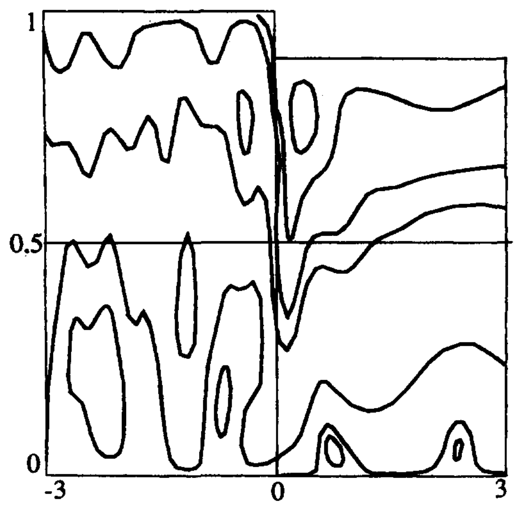

Fig. 4.

By examining the streamlines one can obtain a clear picture of the energy flux and, in some sense, explain the processes that take place in the system. Thus, when the reflection of energy from the interface is insignificant, the streamlines do not form vortices. But when the reflection coefficient is increased, vortices are formed, and they partition off the waveguide.

Figures 2–4 illustrate the pattern of the energy flux when the interfacial boundary has a step of height $0.1h_1$. Here, unlike the case of equal thicknesses, the streamlines are not symmetrical about the longitudinal axis of the waveguide, as the problem cannot be divided into symmetric and antisymmetric problems: a symmetric incident wave will also generate antisymmetric modes in both the reflected and transmitted wave fields. The combination of media was the same as in Fig. 1.

Figure 2 was plotted for a dimensionless frequency of 5.0, that is, before the reflection coefficient begins to increase. At lower frequencies, the streamlines are almost straight lines, flowing smoothly around the step. It is clear from Fig. 3, plotted for a frequency of 5.1, that energy vortices appear as the reflection increases. Figure 4 shows the energy streamlines near maximum reflection, for a dimensionless frequency of 6.0.

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